

# Ma-Xu quantization rule and exact WKB condition for translationally shape invariant potentials

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For translationally shape invariant potentials, the exact quantization rule proposed by Ma and Xu is a direct consequence of exactness of the modified WKB quantization condition proved by Barclay. We propose here a very direct alternative way to calculate the appropriate correction for the whole class of translationally shape invariant potentials.

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## INTRODUCTION

In 2005, Ma and Xu [1, 2], on the basis of a previous work of Cao [3, 4] and al., proposed a new improved quantization rule permitting to retrieve the exact spectrum of some exactly solvable quantum systems. Since, numerous papers have been published on the possible applications of the Ma-Xu formula to different systems [5–14]. Until now, the link between the exactness of the Ma-Xu formula and the solvability of the considered system seems to be stayed unclear. In fact, the translationally shape invariance of the potential suffices to ensure the exactness of the Ma-Xu formula. This is a direct consequence of a result obtained first by Barclay [19] in 1993 and largely overlooked. He established that for these specific potentials the higher order terms in the WKB series can be resummed, yielding to an energy independent correction and he showed that this Maslov index can be obtained in a closed analytical form. More than ten years later, Bhaduri and al. [20] proposed another interesting derivation of this result which rests on periodic orbit theory (POT) [21].

In the present article, after having established the connection between the Ma-Xu formula and Barclay's result, we propose an alternative way to calculate the Maslov index for every translationally shape invariant potential (TSIP). Starting from a classification of TSIP which uses new criterions [22], equivalent to the Barclay-Maxwell ones [18], we show how to obtain this index using simple complex analysis tools.

## MA-XU QUANTIZATION FORMULA

Consider the stationary one dimensional Schrödinger equation ( $\hbar = 1$ ,  $m = 1/2$ ):

$$\psi''(x) + p^2(x)\psi(x) = 0, \quad (1)$$

where  $p(x) = \sqrt{E - V(x)}$  is the classical momentum function for an energy  $E$ . If  $x_1$  and  $x_2$  are the classical turning points then  $p^2(x) \geq 0$  for  $x \in [x_1, x_2]$ .

Defining:

$$w(x) = -\frac{\psi'(x)}{\psi(x)}, \quad (2)$$

we have:

$$w'(x) = -\frac{\psi''(x)}{\psi(x)} + w^2(x).$$

In other words, if  $\psi(x)$  satisfies Eq.(1),  $w(x)$  is a solution of the following Riccati equation:

$$w'(x) = p^2(x) + w^2(x) = E - V(x) + w^2(x). \quad (3)$$

Every node of  $\psi(x)$  (necessarily a simple zero) corresponds to a simple pole of  $w(x)$  which decreases in the interval  $[x_1, x_2]$ . Consequently, when  $x$  runs through  $[x_1, x_2]$ , at each node of  $\psi(x)$  the function  $w(x)$  is subject to a discontinuity from  $-\infty$  to  $+\infty$ . If we define the phase  $\theta(x)$  via:

$$\tan \theta(x) = -\frac{p(x)}{w(x)}, \quad (4)$$

we can write:

$$\theta(x) = \arctan \left( -\frac{p(x)}{w(x)} \right) + n\pi, \quad (5)$$

where  $\arctan y \in [-\pi/2, \pi/2]$  is the principal determination of the reciprocal of the tangent function and where  $n$  increases of 1 at each node of  $w(x)$ .

Then, if  $x_1$  and  $x_2$  are not nodes of  $\psi(x)$ :

$$\int_{x_1}^{x_2} \theta'(x) dx = N\pi, \quad (6)$$

$N$  being the total number of nodes of  $w(x)$  on  $[x_1, x_2]$ .

But using Eq.(3) we also have:

$$\theta'(x) = p(x) - \frac{p'(x)w(x)}{w'(x)}. \quad (7)$$

Considering the  $n^{th}$  bound state at energy  $E_n$  for which  $N_n = n + 1$ , we obtain the following identity:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma(E_n), \quad (8)$$

where the correction  $\gamma(E_n)$  to the lowest order WKB condition is given by the following integral:

$$\gamma(E_n) = \pi + \int_{x_{1,n}}^{x_{2,n}} \frac{w_n(x)p'_n(x)}{w'_n(x)} dx. \quad (9)$$

Ma and Xu [1, 2] have observed that que for a large class of exactly solvable potentials, this integral correction is in fact independent of the considered energy level, and can be calculated from the ground state:

$$\gamma(E_n) = \gamma(E_0) = \pi + \int_{x_{1,0}}^{x_{2,0}} \frac{w_0(x)p'_0(x)}{w'_0(x)} dx. \quad (10)$$

Inserting Eq.(10) in Eq.(8) gives the Ma-Xu formula:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma(E_0). \quad (11)$$

The knowledge of the ground state characteristics permits then to calculate the  $\gamma(E_0)$  correction and reaches to an implicit formula for the other energy levels  $E_n$ . The explicit calculations have been performed for many exactly solvable examples [5–14]. Nevertheless the question of a direct link between the exact solvability of the potential and the validity of the above formula is still stayed open.

As much as we know, all the closed form analytically solvable quantum mechanical systems belongs to the set of translationally shape invariant potentials (TSIP) in the sense of SUSY quantum mechanics [15]. In fact for this set, the validity of the Ma-Xu formula follows from a result established in 1993 by Barclay [19]. In this article, he established

that for these potentials, the WKB series can be resummed beyond the lowest order giving an energy independent correction which can be absorbed in the Maslov index and written in a closed analytical form. He showed equally that this result is directly correlated to the exactness of the lowest order SWKB quantization condition [15, 16]. The starting point is the definition of two classes of potentials, each characterized by a specific change of variable which brings the potential into a quadratic form. This two classes are shown to coincide to the Barclay-Maxwell classes [18] which are based on a functional characterization of superpotentials and which cover the whole set of TSIP. In both cases, the action variable for an energy  $E_n$  is of the form:

$$I_n = \oint_{E_n} p_n(x) dx = 2 \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = 2(n\pi + \gamma), \quad (12)$$

where  $\gamma$  is an energy-independent correction characteristic of the considered potential. As shown later by Bhaduri and al. [20], this results can be retrieved in an elegant way using POT [21].

The translational shape invariance of the potential is then a sufficient condition for the validity of the Ma-Xu prescription which reaches to Eq.(11).

In the following, we propose an alternative way to recover Barclay's result. It lies on a different characterization of the Barclay-Maxwell classes that we recently presented [22]. In both cases the action variable can be rewritten as a complex integral on an uniformization domain including only one branch-cut. Using standard tools of complex analysis, it can be readily calculated to recover the explicit form of  $\gamma$  for the whole set of TSIP.

Note that analogous complex analysis techniques have been employed by Bhalla and al [24–26] in the frame of the quantum Hamilton-Jacobi (QHJ) formalism [28, 29] where, starting from the exact QHJ quantization condition, they have computed the spectrum of numerous potentials for which they have also showed the exactness of the lowest order SWKB quantization condition. Moreover, Cherqui et al [27] have recently shown that for algebraic and hyperbolic translationally shape invariant potentials, the shape invariance condition provides sufficient information on the singularity structure of the quantum momentum function to determine directly the energy spectrum of the system from the exact QHJ quantization condition.

## FIRST CATEGORY POTENTIALS

We say that a one dimensional potential is of first category [22] if there exists a change of variable  $x \rightarrow u$  transforming the potential into an harmonic one  $V(x) \rightarrow V(u) = \tilde{\lambda}_2 u^2 + \tilde{\lambda}_1 u + \tilde{\lambda}_0$ , such that  $u(x)$  satisfy a constant coefficient Riccati equation:

$$\frac{du(x)}{dx} = A_0 + A_1 u(x) + A_2 u^2(x), \quad (13)$$

$du/dx$  being of constant sign in all the range of values of  $x$  and  $u$ .

The one dimensional harmonic oscillator correspond to the special case  $A_1 = A_2 = 0$  and the Morse potential is generically associated to the case  $A_1 \neq 0$ ,  $A_2 = 0$ .

### Harmonic oscillator

It is perfectly well known that the EBK quantization condition is exact for the harmonic oscillator implying then the exactness of the Ma-Xu formula with a Maslov index equal to  $1/2$ . Nevertheless, we will examine this case completely as a first example of use of the complex variable integration technique.

The harmonic oscillator potential with zero ground state energy is:

$$V(x, \omega) = (\omega/2)^2 x^2 - \omega/2. \quad (14)$$

The classical turning points  $\pm x_{0,n}$  at energy  $E_n = n\omega$  satisfy:

$$x_{0,n}^2 = \frac{4}{\omega} \left( n + \frac{1}{2} \right). \quad (15)$$

The half action variable for a classical periodic orbit of energy  $E_n$  is then:

$$I_n = \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = \frac{\omega}{2} \int_{-x_{0,n}}^{x_{0,n}} dx \sqrt{x_{0,n}^2 - x^2}. \quad (16)$$

The complex function  $f_H(z) = \sqrt{x_{0,n}^2 - z^2}$ , defined on  $\mathbb{C} - ]-x_{0,n}, x_{0,n}[$ , admits only one isolated singularity at infinity. Then:

$$\int_{-x_{0,n}}^{x_{0,n}} dx f_H(x) = \pi i \text{Res}(f_H(z), \infty). \quad (17)$$

The asymptotic behaviour at infinity on the Riemann sheet for which  $\sqrt{u_{0,n}^2 - z^2} = \sqrt{x_{0,n}^2 + y^2} > 0$  on the positive half imaginary axe ( $z = iy, y > 0$ ) is given by:

$$\sqrt{x_{0,n}^2 - z^2} \underset{z \rightarrow \infty}{\sim} -iz \left(1 - \frac{x_{0,n}^2}{z^2}\right)^{\frac{1}{2}} \underset{z \rightarrow \infty}{\sim} -iz + \frac{ix_{0,n}^2}{2} \frac{1}{z} + O\left(\frac{1}{z^2}\right). \quad (18)$$

Consequently:

$$\text{Res}(f_H(z), \infty) = -\frac{ix_{0,n}^2}{2} \quad (19)$$

and:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = \left(n + \frac{1}{2}\right) \pi. \quad (20)$$

which is the expected result.

### Morse potential

As for the harmonic oscillator, the EBK formula is exact for the Morse potential. This second example is nevertheless very instructive. The Morse potential with zero energy ground state is [15]:

$$V(x) = A^2 + B^2 e^{-2\alpha x} - 2B \left(A + \frac{\alpha}{2}\right) e^{-\alpha x}, \quad \alpha > 0. \quad (21)$$

Using the change of variable  $y = \exp(-\alpha x)$ ,  $dy = -\alpha y dx$ , it becomes:

$$V(y) = B^2 (y - y_0)^2 + V_0 \quad (22)$$

with  $y_0 = (A + \alpha/2)/B$  and:

$$V_0 = A^2 - B^2 y_0^2 = \left(A + \frac{\alpha}{2}\right)^2 - A^2. \quad (23)$$

This is a translationally shape invariant potential with a spectrum given by [22]:

$$E_n = a^2 - a_n^2. \quad (24)$$

with  $a = A$  and  $a_k = A - k\alpha$ .

As for the classical turning points, they are given by:

$$\begin{cases} y_{2,n} = y_0 + u_{0,n} \\ y_{1,n} = y_0 - u_{0,n} \end{cases} \quad (25)$$

with:

$$u_{0,n} = \sqrt{y_0^2 - \frac{a_n^2}{B^2}}. \quad (26)$$

The half action variable for a classical periodic orbit of energy  $E_n$  is then:

$$\begin{aligned} \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx &= -\frac{1}{\alpha} \int_{y_{1,n}}^{y_{2,n}} \sqrt{E_n - V(y, a)} \frac{dy}{y} \\ &= \frac{B}{\alpha} \int_{-u_{0,n}}^{u_{0,n}} f_M(u) du, \end{aligned} \quad (27)$$

where:

$$f_M(u) = \frac{\sqrt{u_{0,n}^2 - u^2}}{u + y_0}. \quad (28)$$

The complex extended function  $f_M(z) = \sqrt{u_{0,n}^2 - z^2} / (z + y_0) T$ , defined on  $\mathbb{C} - ]-u_{0,n}, u_{0,n}[$ , admits two isolated singularities at infinity and in  $-y_0$ . Then:

$$\int_{-u_{0,n}}^{u_{0,n}} du f_M(u) = \pi i (\text{Res}(f_M(z), -y_0) + \text{Res}(f_M(z), \infty)). \quad (29)$$

With the same choice as before for the square root determination we have:

$$f_M(z) \underset{z \rightarrow \infty}{\sim} i \left( -1 + \frac{y_0}{z} + O\left(\frac{1}{z^2}\right) \right) \quad (30)$$

which implies:

$$\text{Res}(f_M(z), \infty) = -i \frac{a + \alpha/2}{B}. \quad (31)$$

The residue in  $-y_0$  (which is a simple pole) is readily obtained as:

$$\text{Res}(f_M(z), -y_0) = i \frac{a - n\alpha}{B}. \quad (32)$$

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = \left( n + \frac{1}{2} \right) \pi. \quad (33)$$

which is the expected result.

### Effective radial potential for the Kepler-Coulomb problem

Consider the effective radial potential for the Kepler-Coulomb. It can be written as [22]:

$$V(y) = l(l+1)y^2 - \gamma y + \frac{\gamma^2}{4(l+1)^2}, \quad k > 0, \quad (34)$$

where  $y = 1/x$  and  $dy = -y^2 dx$ . We can rather use the equivalent form:

$$V(y) = l(l+1)(y - y_0)^2 + V_0, \quad (35)$$

with  $y_0 = \gamma/2l(l+1)$  and

$$V_0 = \frac{\gamma^2}{4(l+1)^2} - \frac{\gamma^2}{4l(l+1)}. \quad (36)$$

This is a translationally shape invariant potential characterized by the following energy spectrum [22]:

$$E_n(a) = \frac{\gamma^2}{4a^2} - \frac{\gamma^2}{4a_n^2}. \quad (37)$$

where  $a = l+1$  and  $a_k = l+1+k$ .

The classical turning points are given by:

$$\begin{cases} y_{2,n} = y_0 + u_{0,n} \\ y_{1,n} = y_0 - u_{0,n} \end{cases} \quad (38)$$

with:

$$u_{0,n} = \frac{1}{\sqrt{l(l+1)}} \sqrt{\frac{\gamma^2}{4l(l+1)} - \frac{\gamma^2}{4a_n^2}}. \quad (39)$$

The half action variable for a classical periodic orbit of energy  $E_n$  is then:

$$\begin{aligned} \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx &= - \int_{y_{1,n}}^{y_{2,n}} \sqrt{E_n - V(y, a)} \frac{dy}{y^2} \\ &= -\sqrt{l(l+1)} \int_{-u_{0,n}}^{u_{0,n}} du f_K(u), \end{aligned} \quad (40)$$

where:

$$f_K(u) = \frac{\sqrt{u_{0,n}^2 - u^2}}{(u + y_0)^2}. \quad (41)$$

Again, the complex function  $f_K(z) = \sqrt{u_{0,n}^2 - z^2} / (z + y_0)^2$ , defined on  $\mathbb{C} - ]-u_{0,n}, u_{0,n}[$ , admits two isolated singularities at infinity and in  $-y_0$ . Then:

$$\int_{-u_{0,n}}^{u_{0,n}} du f_K(u) = \pi i (\text{Res}(f_K(z), -y_0) + \text{Res}(f_K(z), \infty)). \quad (42)$$

With the same choice as before for the square root determination, we have:

$$f_K(z) \underset{z \rightarrow \infty}{\sim} \frac{-i}{z} \left(1 - \frac{u_{0,n}^2}{z^2}\right)^{\frac{1}{2}} \left(1 + \frac{y_0}{z}\right)^{-2} \underset{z \rightarrow \infty}{\sim} \frac{-i}{z} + O\left(\frac{1}{z^2}\right) \quad (43)$$

which implies:

$$\text{Res}(f_K(z), \infty) = i. \quad (44)$$

The residue in  $-y_0$  which is a double pole is readily obtained as:

$$\text{Res}(f_K(z), -y_0) = i \frac{a_n}{\sqrt{l(l+1)}}. \quad (45)$$

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma, \quad (46)$$

where:

$$\gamma = \pi \left( l + 1 + \sqrt{l(l+1)} \right)$$

which is the expected result.

### Other first category potentials

If we except the two preceding examples, all the other first category potentials are such that there exists a change of variable  $x \rightarrow y$  transforming the potential into an harmonic one:

$$V(x) \rightarrow V(y) = a(a \mp \alpha)y^2 + \lambda_1 y + \lambda_0(a),$$

with  $\lambda_0(a) = \lambda_1^2/4a^2 - \alpha a$  (in order to have a zero energy ground state) and where  $y(x)$  satisfies a constant coefficient Riccati equation of the form [22]:

$$\frac{dy}{dx} = \alpha \pm \alpha y^2(x) > 0. \quad (47)$$

The potential can still be written [22]:

$$V_{\pm}(y, a) = a(a \mp \alpha)(y - y_0)^2 + V_0, \quad (48)$$

where:

$$y_0 = -\lambda_1/a(a \mp \alpha), \quad V_0 = \lambda_0(a) - a(a \mp \alpha)y_0^2 \quad (49)$$

$V$  is a translationally shape invariant potential and its energy spectrum is then given by [22]:

$$E_n(a) = \phi_{1,\pm}(a) - \phi_{1,\pm}(a_n), \quad (50)$$

where  $a_k = a \pm k\alpha$  and:

$$\phi_{1,\pm}(a) = \mp a^2 + \frac{\lambda_1^2}{4a^2}. \quad (51)$$

The classical turning points  $y_{i,n}$  for an energy  $E_n$  are determined by the condition:

$$E_n = V(y_{i,n}) \Leftrightarrow E_n = a(a \mp \alpha)u_{0,n}^2 + V_0, \quad (52)$$

where:

$$\begin{cases} y_{2,n} = y_0 + u_{0,n} \\ y_{1,n} = y_0 - u_{0,n} \end{cases}, \quad (53)$$

and:

$$u_{0,n} = \sqrt{\mp 1 + y_0^2 - \frac{\phi_{1,\pm}(a_n)}{a(a \mp \alpha)}}. \quad (54)$$

Consider first the case where  $dy/dx = \alpha + \alpha y^2(x)$ . The half action variable for a classical periodic orbit of energy  $E_n$  is given by:

$$\begin{aligned} \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx &= \int_{y_{1,n}}^{y_{2,n}} \sqrt{E_n - V(y, a)} \frac{dy}{\alpha(y-i)(y+i)} \\ &= \frac{\sqrt{a(a-\alpha)}}{\alpha} \int_{-u_{0,n}}^{u_{0,n}} f_{1,+}(u) du, \end{aligned} \quad (55)$$

where:

$$f_{1,+}(z) = \frac{\sqrt{u_{0,n}^2 - z^2}}{(z - z_0)(z - \bar{z}_0)} \quad (56)$$

with  $z_0 = -y_0 + i$ .

The complex extended integrand  $f_{1,+}(z)$ , defined on  $\mathbb{C} - ]-u_{0,n}, u_{0,n}[$ , admits three isolated singularities at infinity,  $z_0$  and  $\bar{z}_0$ . Then:

$$\int_{-u_{0,n}}^{u_{0,n}} du f_{1,+}(u) = \pi i (\text{Res}(f_{1,+}(z), z_0) + \text{Res}(f_{1,+}(z), \bar{z}_0) + \text{Res}(f_{1,+}(z), \infty)). \quad (57)$$

When  $z \rightarrow \infty$ :

$$f_{1,+}(z) \underset{z \rightarrow \infty}{\sim} -\frac{i}{z} + O\left(\frac{1}{z^2}\right), \quad (58)$$

which implies:

$$\text{Res}(f_{1,+}(z), \infty) = i. \quad (59)$$

As for the residues in  $z_0$  and  $\bar{z}_0$ , using Eq.(52), Eq.(53), Eq.(54) and Eq.(50) we deduce:

$$\text{Res}(f_{1,+}(z), z_0) = -\frac{1}{2\sqrt{a(a-\alpha)}} \left( ia_n + \frac{\lambda_1}{2a_n} \right) \quad (60)$$



and:

$$\text{Res}(f_{1,+}(z), \bar{z}_0) = -\frac{1}{2\sqrt{a(a-\alpha)}} \left( -ia_n + \frac{\lambda_1}{2a_n} \right). \quad (61)$$

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma \quad (62)$$

with:

$$\gamma = \frac{\pi a}{\alpha} \left( 1 - \sqrt{1 - \frac{\alpha}{a}} \right). \quad (63)$$

Consider now the case  $dy/dx = \alpha - \alpha y^2(x)$ . The half action variable for a classical periodic orbit of energy  $E_n$  is:

$$\begin{aligned} \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx &= \int_{y_{1,n}}^{y_{2,n}} \sqrt{E_n - V(y, a)} \frac{dy}{\alpha(1-y)(1+y)} \\ &= -\frac{\sqrt{a(a-\alpha)}}{\alpha} \int_{-u_{0,n}}^{u_{0,n}} f_{1,-}(u) du, \end{aligned} \quad (64)$$

where:

$$f_{1,-}(u) = \frac{\sqrt{u_{0,n}^2 - u^2}}{(u - z_1)(u - z_2)} \quad (65)$$

with  $z_1 = -1 - y_0$  and  $z_2 = 1 - y_0$ .

The complex extended integrand  $f_{1,+}(z)$ , defined on  $\mathbb{C} - ]-u_{0,n}, u_{0,n}[$ , admits three isolated singularities at infinity,  $z_1$  and  $z_2$ . Then:

$$\int_{-u_{0,n}}^{u_{0,n}} du f_{1,-}(u) = \pi i (\text{Res}(f_{1,-}(z), z_1) + \text{Res}(f_{1,-}(z), z_2) + \text{Res}(f_{1,-}(z), \infty)). \quad (66)$$

When  $z \rightarrow \infty$ :

$$f_{1,+}(z) \underset{z \rightarrow \infty}{\sim} -\frac{i}{z} + O\left(\frac{1}{z^2}\right), \quad (67)$$

which implies:

$$\text{Res}(f_{1,+}(z), \infty) = i. \quad (68)$$

As for the residues in  $z_0$  and  $\bar{z}_0$ , from Eq.(52), Eq.(91), Eq.(92) and Eq.(50) we deduce:

$$\text{Res}(f_{1,-}(z), z_2) = \frac{i}{2\sqrt{a(a-\alpha)}} \left( a_n + \frac{\lambda_1}{2a_n} \right) \quad (69)$$

and:

$$\text{Res}(f_{1,-}(z), z_1) = \frac{i}{2\sqrt{a(a-\alpha)}} \left( a_n - \frac{\lambda_1}{2a_n} \right). \quad (70)$$

Note that in this case, we had to change the square root determination when we pass from  $z_1$  to  $z_2$ , since we have either  $-1 < y_{1,n} < y_{2,n} < 1$  (when  $y = \tanh(\alpha x + \varphi)$ ), that is  $-1 - y_0 < -u_{0,n} < u_{0,n} < 1 - y_0$ , or  $y_{1,n} < -1 < 1 < y_{2,n}$  (when  $y = \coth(\alpha x + \varphi)$ ), that is  $-u_{0,n} < -1 - y_0 < 1 - y_0 < u_{0,n}$ .

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma \quad (71)$$

with:

$$\gamma = -\frac{\pi a}{\alpha} \left( 1 + \sqrt{1 - \frac{\alpha}{a}} \right). \quad (72)$$

We recover the results obtained in [20] for the first class potentials.

## SECOND CATEGORY POTENTIALS

We say that a one dimensional potential is of second category if there exists a change of variable  $x \rightarrow u$  transforming the potential into an isotonic one  $V(x) \rightarrow V(u) = \tilde{\lambda}_2 u^2 + \tilde{\lambda}_0 + \frac{\tilde{\mu}_2}{u^2}$ , such that  $u(x)$  satisfies a constant coefficients Riccati equation of the form:

$$\frac{du(x)}{dx} = A_0 + A_2 u^2(x), \quad (73)$$

$du(x)/dx$  being of constant sign in all the range of values of  $x$  and  $u$ .

### Isotonic oscillator

The isotonic potential with a zero energy ground state ( $E_0 = 0$ ) is [22, 30]:

$$V_-(x) = \frac{\omega^2}{4} x^2 + \frac{l(l+1)}{x^2} - \omega \left( l + \frac{3}{2} \right), \quad l > 0. \quad (74)$$

Its energy spectrum is given by [22]:

$$E_n(a) = 2n\omega, \quad (75)$$

where  $a = (\frac{\omega}{2}, l+1)$  and  $a_k = (\frac{\omega}{2}, l+1+k)$ .

The classical turning points  $x_{i,n}$  for an energy  $E_n$  are determined by the condition:

$$E_n = V(x_{i,n}) \Leftrightarrow \frac{\omega^2}{4} x_{i,n}^4 - \omega \left( 2n + l + \frac{3}{2} \right) x_{i,n}^2 + l(l+1) = 0, \quad (76)$$

that is:

$$\begin{cases} x_{2,n}^2 = u_{0,n} + \delta_n \\ x_{1,n}^2 = u_{0,n} - \delta_n \end{cases} \quad (77)$$

with:

$$u_{0,n} = 2 \frac{2n + l + \frac{3}{2}}{\omega}, \quad \delta_n = \frac{2}{\omega} \sqrt{\left( 2n + l + \frac{3}{2} \right)^2 - l(l+1)}. \quad (78)$$

The half action variable for a classical periodic orbit of energy  $E_n$  is then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = \frac{\omega}{4} \int_{x_{1,n}^2}^{x_{2,n}^2} du f_I(u),$$

where we have defined  $u = x^2$  and:

$$f_I(u) = \frac{\sqrt{(u - x_{1,n}^2)(x_{2,n}^2 - u)}}{u}.$$

The complex extended function  $f_I(z) = \sqrt{(z - x_{1,n}^2)(x_{2,n}^2 - z)}/z$ , defined on  $\mathbb{C} - ]x_{1,n}^2, x_{2,n}^2[$ , admits two isolated singularities at infinity and in 0. Then:

$$\int_{x_{1,n}^2}^{x_{2,n}^2} du f_I(u) = \pi i (\text{Res}(f_I(z), 0) + \text{Res}(f_I(z), \infty)). \quad (79)$$

When  $z \rightarrow \infty$ :

$$f_I(z) \underset{z \rightarrow \infty}{\sim} -i \left( 1 - \frac{x_{2,n}^2 + x_{1,n}^2}{2z} + O\left(\frac{1}{z^2}\right) \right), \quad (80)$$

which implies:

$$\text{Res}(f_I(z), \infty) = -iu_{0,n} = -2i \frac{2n + l + \frac{3}{2}}{\omega^2}. \quad (81)$$

As for the residue in 0, from Eq.(52) we have:

$$\text{Res}(f_I(z), 0) = \sqrt{-x_{1,n}^2 x_{2,n}^2} = i \frac{2}{\omega} \sqrt{l(l+1)}. \quad (82)$$

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma, \quad (83)$$

with:

$$\gamma = \frac{\pi}{2} \left( l + \frac{3}{2} - \sqrt{l(l+1)} \right). \quad (84)$$

### Other second category potentials

If we except the preceding example, all the other second category potentials are such that there exists a change of variable  $x \rightarrow y$  transforming the potential into an isotonic one  $V(x) \rightarrow V(y) = \lambda_2 y^2 + \frac{\mu_2}{y^2} + \lambda_0$ , where  $y(x) > 0$  satisfies a constant coefficient Riccati equation of the form [22]:

$$\frac{dy}{dx} = \alpha \pm \alpha y^2(x), \quad (85)$$

$dy(x)/dx$  being of constant sign in all the range of values of  $x$  and  $y$ .

As shown in [22], the potential can be written:

$$V(y, a) = \lambda (\lambda \mp \alpha) y^2 + \frac{\mu (\mu - \alpha)}{y^2} + \lambda_0(a), \quad (86)$$

where  $a = (\lambda, \mu)$  and:

$$\lambda_0(a) = -\alpha (\lambda \pm \mu) - 2\lambda\mu. \quad (87)$$

$V$  is a translationally shape invariant potential and its energy spectrum is given by [22]:

$$E_n(a) = \pm (\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a)), \quad (88)$$

where  $a = a_0$ ,  $a_k = (\lambda_k, \mu_k) = (\lambda \pm k\alpha, \mu + k\alpha)$  and:

$$\phi_{2,\pm}(a) = \phi_{2,\pm}(\lambda, \mu) = (\lambda \pm \mu)^2. \quad (89)$$

The classical turning points  $y_{i,n}$  for an energy  $E_n$  are determined by the condition:

$$E_n = V(y_{i,n}) \Leftrightarrow \lambda (\lambda \mp \alpha) y_{i,n}^4 + (\lambda_0(a) - E_n) y_{i,n}^2 + \mu (\mu - \alpha) = 0, \quad (90)$$

that is:

$$\begin{cases} y_{2,n}^2 = u_{0,n} + \delta_n \\ y_{1,n}^2 = u_{0,n} - \delta_n \end{cases} \quad (91)$$

with:

$$u_{0,n} = \frac{E_n - \lambda_0(a)}{2\lambda(\lambda \mp \alpha)}, \quad \delta_n = \sqrt{\left(\frac{E_n - \lambda_0(a)}{2\lambda(\lambda \mp \alpha)}\right)^2 - \frac{\mu(\mu - \alpha)}{\lambda(\lambda \mp \alpha)}}. \quad (92)$$

In the case  $dy/dx = \alpha + \alpha y^2$ , the half action variable for a classical periodic orbit of energy  $E_n$  is:

$$\begin{aligned} \int_{x_{1,n}}^{x_{2,n}} p_n(x) dx &= \int_{y_{1,n}}^{y_{2,n}} \sqrt{E_n - V(y, a)} \frac{dy}{\alpha(y^2 + 1)} \\ &= \frac{\sqrt{\lambda(\lambda - \alpha)}}{2\alpha} \int_{y_{1,n}^2}^{y_{2,n}^2} du f_{2,+}(u), \end{aligned} \quad (93)$$

where we have defined  $u = y^2$  and:

$$f_{2,+}(z) = \frac{\sqrt{(z - y_{1,n}^2)(y_{2,n}^2 - z)}}{z(z + 1)}. \quad (94)$$

This last integral is readily calculated by using a complex variable formalism. We have indeed:

$$\int_{y_{1,n}^2}^{y_{2,n}^2} du f_{2,+}(u) = \pi i (\text{Res}(f_{2,+}(z), 0) + \text{Res}(f_{2,+}(z), -1) + \text{Res}(f_{2,+}(z), \infty)). \quad (95)$$

When  $z \rightarrow \infty$ :

$$f_{2,+}(z) \underset{z \rightarrow \infty}{\sim} -\frac{i}{z} + O\left(\frac{1}{z^2}\right), \quad (96)$$

which implies:

$$\text{Res}(f_{2,+}(z), \infty) = i. \quad (97)$$

As for the residues in 0 and  $-1$ , from Eq.(90) we have:

$$\text{Res}(f_{2,+}(z), 0) = \sqrt{-y_{1,n}^2 y_{2,n}^2} = i \sqrt{\frac{\mu(\mu - \alpha)}{\lambda(\lambda - \alpha)}}. \quad (98)$$

Using Eq.(90), Eq.(91), Eq.(92) and Eq.(88), we deduce  $(-1 < 0 < y_{1,n}^2 < y_{2,n}^2)$ :

$$\text{Res}(f_{2,+}(z), -1) = -\sqrt{-(1 + y_{1,n}^2)(y_{2,n}^2 + 1)} = \frac{-i(\lambda_n + \mu_n)}{\sqrt{\lambda(\lambda - \alpha)}}. \quad (99)$$

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma \quad (100)$$

with:

$$\gamma = \frac{\pi\lambda}{2\alpha} \left(1 - \sqrt{1 - \frac{\alpha}{\lambda}}\right) + \frac{\pi\mu}{2\alpha} \left(1 - \sqrt{1 - \frac{\alpha}{\mu}}\right). \quad (101)$$

A similar analysis can be led in the case  $dy/dx = \alpha - \alpha y^2(x)$ . If we except the specific case of the Scarf II potential [22], we can write:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = -\frac{\sqrt{\lambda(\lambda + \alpha)}}{2\alpha} \int_{y_{1,n}^2}^{y_{2,n}^2} du f_{2,-}(u), \quad (102)$$

where we have defined  $u = y^2$  and:

$$f_{2,-}(z) = \frac{\sqrt{(z - y_{1,n}^2)(y_{2,n}^2 - z)}}{z(z - 1)}. \quad (103)$$

We have:

$$\int_{y_{1,n}^2}^{y_{2,n}^2} du f_{2,-}(u) = \pi i (\text{Res}(f_{2,-}(z), 0) + \text{Res}(f_{2,-}(z), 1) + \text{Res}(f_{2,-}(z), \infty)). \quad (104)$$

When  $z \rightarrow \infty$ :

$$f_{2,-}(z) \underset{z \rightarrow \infty}{\sim} -\frac{i}{z} + O\left(\frac{1}{z^2}\right), \quad (105)$$

which implies:

$$\text{Res}(f_{2,-}(z), \infty) = i. \quad (106)$$

As for the residues in 0 and 1, using Eq.(90), Eq.(91), Eq.(92) and Eq.(88), we deduce:

$$\text{Res}(f_{2,-}(z), 0) = -\sqrt{-y_{1,n}^2 y_{2,n}^2} = -i\sqrt{\frac{\mu(\mu - \alpha)}{\lambda(\lambda + \alpha)}} \quad (107)$$

and

$$\text{Res}(f_{2,-}(z), 1) = \sqrt{(1 - y_{1,n}^2)(y_{2,n}^2 - 1)} = \frac{-i(\lambda_n - \mu_n)}{\sqrt{\lambda(\lambda + \alpha)}}. \quad (108)$$

Note that in this case, we had to change the square root determination when we pass from 0 to 1, since we have  $0 < y_{1,n}^2 < y_{2,n}^2 < 1$  ( $y = \tanh(\alpha x + \varphi)$ ).

Then:

$$\int_{x_{1,n}}^{x_{2,n}} p_n(x) dx = n\pi + \gamma \quad (109)$$

with:

$$\gamma = \frac{\pi\mu}{2\alpha} \left(1 - \sqrt{1 - \frac{\alpha}{\mu}}\right) - \frac{\pi\lambda}{2\alpha} \left(1 - \sqrt{1 + \frac{\alpha}{\lambda}}\right). \quad (110)$$

In the case of the Scarf II potential, the variable  $y(x) = \tanh(\alpha x/2 + i\pi/4)$  is a pure phase factor ( $|y| = 1$ ) and the path of integration in the action variable surrounds the branch cut which is an arc of the unit circle. Nevertheless, we can follow the same reasoning as before and we recover the result given in Eq.(110).

Eq.(101) and Eq.(110) correspond to the results obtained in [20] for the second class potentials.

## CONCLUSION

We have shown how to calculate exactly and in a general way the action variable for the whole set of translationally shape invariant potentials. The correction to lowest order WKB formula appears as a constant term redefining the Maslov index. This result implies immediately the exactness of the Ma-Xu formula for every TSIP. The employed techniques of complex analysis are standard but even so instructive. The two basic cases are the harmonic and isotonic ones (for which we have an equispaced quantum spectrum and isochronicity at the classical level). In this two cases, the action integral involved is a sum of at most two residues and the linear  $n$  dependence ( $n$  being the energy quantum number) comes from the residue at infinity. For all the other cases, the integral involved is obtained from the two basic cases by deforming the integration measure with a weight which is the inverse of an at most quadratic polynomial. The residue at infinity becomes energy independent and the linear  $n$  dependence takes its origin in the finite poles (with an eventual compensation of nonlinear  $n$  dependent contributions).

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